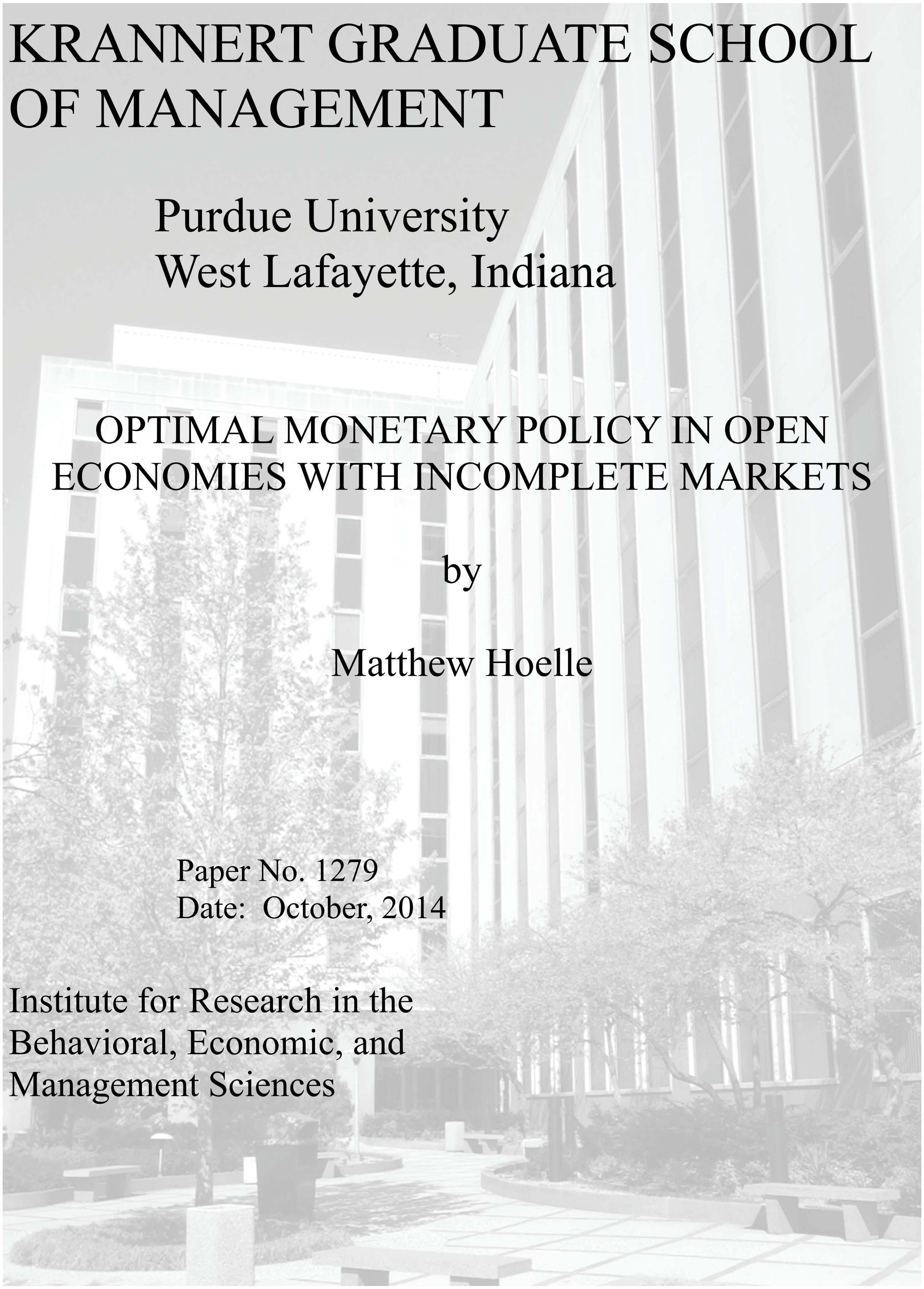

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OPTIMAL MONETARY POLICY IN OPEN ECONOMIES WITH INCOMPLETE MARKETS

by

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Optimal Monetary Policy in Open Economies with Incomplete Markets*

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Abstract

In stochastic settings of large open economies, the monetary policies of central banks have real effects. This paper considers a simple N-country model with homogeneous households in each country and conventional central bank policy (open market operations). The conditions for Pareto efficiency are characterized. Symmetric policies, or even policies in which both countries adopt some form of targeting rule, are not consistent with Pareto efficiency (generically).

Keywords monetary policy rules – exchange rate – inflation rate targeting – nominal GDP targeting

JEL Classification **D50, E31, E40, E50, F41**

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1 Introduction

In an international setting with risk, the policies of central banks have real effects on the equilibrium allocation. This occurs because households trade the government debt issued

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by the fiscal authorities (and targeted by the central banks) in these countries and markets are incomplete. In this model, the only friction is incomplete markets, namely a deficient number of financial assets to cover all risky outcomes. A change in central bank policy changes the real asset payouts faced by households, both domestic and foreign, leading to welfare effects for all households. This paper introduces a general equilibrium framework that captures these mechanisms and analyzes the restrictions imposed on any policies consistent with Pareto efficiency.

This paper considers a large open economy with N countries. Household heterogeneity only occurs across countries and not within a country, as I assume that each country is inhabited by a unit mass of homogeneous households. Risk can include both idiosyncratic risk and aggregate risk. The idiosyncratic risk faced by any household is stochastic endowment (output) risk. Aggregate risk is also permitted, meaning that the total world endowment can fluctuate over states of uncertainty.

The fiscal authority in each country issues nominally risk-free, short-term government debt, which I simply call bonds. Given the presence of government bonds, the central banks choose their portfolios (including both domestic and foreign government bonds) in order to support a particular money supply and price level. Without any role for fiscal policy, the fiscal authorities are omitted in this model. The number of domestic bonds issued by the fiscal authority in any country is simply included in the portfolio choice of the domestic central bank. A country's bonds can be held by households (both domestic and foreign) and central banks (both domestic and foreign). I impose the natural restriction that a central bank cannot hold more domestic bonds than were initially issued by the domestic fiscal authority, meaning that the net debt position is nonnegative.

With N countries, there are N assets available for trade, namely a bond issued by each country. Households can buy or sell these assets, subject only to an implicit debt constraint. Incomplete markets requires that the number of states of uncertainty must be strictly larger than N, the number of assets.

Central banks choose the money supplies and debt positions in each period in order to target the interest rates on the domestic bond. Money is valued in this model via a cash-in-advance constraint, which implies a linear relation between the money supply and the price level in each country. Thus, although the bond is nominally risk-free, the payoff in real terms (the relevant payoff from the households' points of view) is endogenously determined by the monetary policy.

A Markov equilibrium is introduced and is shown to be the most general equilibrium concept that is both recursive and a subset of the larger class of sequential competitive equilibria. I prove that a Markov equilibrium always exists. The equilibrium is specified for all possible policy choices of the central bank, and not just those that support a Pareto efficient allocation.

I then characterize the necessary conditions under which a Pareto efficient allocation can be supported. These conditions are generic, meaning that they are necessary for an open and full measure subset of household endowments. The conditions depend upon the number of states of uncertainty. Pareto efficiency is not possible if all countries adopt inflation rate targets, regardless of the value of these targets. The same holds true for interest rate targeting. If all countries adopt monetary growth targets, or equivalently nominal GDP targets, then the necessary condition for Pareto efficiency is $S \leq N + 1$, where S is the number of states of uncertainty and N is the number of countries.

Symmetric policies of inflation rate targeting, where the targets can vary by country, are optimal policies in New Keynesian models of closed economies, in which the frictions are nominal rigidities and monopolistic competition. Optimal monetary policy has zero or close to zero inflation in order to mitigate the effects of the nominal rigidities.[1] In open economies, optimal policies require policy makers to trade off exchange rate stabilization (to account for the nominal rigidities being imported from foreign firms) with domestic price stabilization (to simultaneously minimize the domestic output gap and the effect of domestic nominal distortions).[2]

Symmetric policies of nominal GDP targeting, where the targets can vary by country, are optimal policies in the closed economy model in Koenig (2013) and the open economy model in Sheedy (2014). Both analyses model incomplete markets as the main friction that introduces a role for monetary policy. Koenig (2013) considers a stylized 2-period setting. Sheedy (2014) only considers aggregate risk, and assumes a special structure on the preferences in order to ensure a stationary wealth distribution. In a standard incomplete markets model with a finite time horizon (and a well-defined terminal condition), nominal GDP targeting rules are optimal policies, meaning that they support Pareto efficient equilibrium allocations.[3] However, this paper proves that it is only possible for nominal GDP targeting to support a Pareto efficient equilibrium allocation when $S \leq N + 1$, where S is the number of states of uncertainty and N is the number of countries. The result holds regardless of the type of risk in the economy: idiosyncratic, aggregate, or both.

Even for such economies, the nominal GDP targets themselves cannot be set by the

[1] A representative sample of the important papers using this class of models in a closed economy include Galí (1992), Sims (1992), Bernanke and Mihov (1998), Christiano et al. (1999), Taylor (1999), Clarida et al. (2000), Woodford (2003), Schmitt-Grohé and Uribe (2004), Uhlig (2005), and the recent Gertler and Karadi (2011) that extend the business cycle models of Christiano et al. (2005) and Smets and Wouters (2007).

[2] See Benigno and Benigno (2003), Corsetti and Pesenti (2005), Devereaux and Sutherland (2008), Corsetti et al. (2008b), and Corsetti et al. (2010).

[3] The argument is provided in Hoelle and Peiris (2013).

central banks, but must be variables. There is no guarantee that the target values consistent with Pareto efficiency are feasible and reasonable choices for the central banks.[4]

The key for the Pareto efficiency result is to maintain a stationary wealth distribution. This is essential for models with infinite-lived households. This paper shows in a general setting that a restrictive upper bound on the number of states of uncertainty is required for a stationary wealth distribution. Sheedy (2014) is able to ensure a stationary wealth distribution by making an assumption involving a time-varying discount factor. The only other option is to consider a model with finite-lived households, such as an overlapping generations model. However, as demonstrated by Kehoe and Levine (1990) and Feng (2013), OLG models with incomplete markets (typically referred to as stochastic OLG models, or SOLG models) contain an inherent real indeterminacy. Without a well-defined mapping from policy to equilibrium allocation, any conclusions about the effects of policy on *one* equilibrium (rather than on the entire equilibrium set) are irrelevant. The present cash-in-advance model exhibits real determinacy and avoids this problem.

The remainder of the paper is organized as follows. Section 2 connects the findings of this paper with the broad literature in the fields of macroeconomics, international economics, and international finance. Section 3 introduces the model. Section 4 provides the recursive equilibrium definition. Section 5 characterizes the necessary conditions for Pareto efficiency. Section 6 provides concluding remarks and the proofs of the main results are contained in Appendix A.

2 Literature Review

The research question in the present work is the extent to which monetary policy, which causes exchange rate movements in an international setting, can mitigate the real effects of incomplete financial markets.

In international economics, the typical friction is nominal rigidities. Models with such frictions belong to the New Keynesian tradition. Notable exceptions include Baxter and Crucini (1995) and Corsetti et al. (2008a). Baxter and Crucini (1995) analyze the effect of incomplete markets on international business cycles. They find that if the idiosyncratic output risk is small, the presence of incomplete markets does not have large real effects. Corsetti et al. (2008a) extends this result by showing that terms of trade volatility magnifies the effects from Baxter and Crucini (1995), meaning that exchange rate fluctuations (caused by monetary policy) allow incomplete markets to play a greater role in business cycle

[4] Feasibility refers to the equilibrium requirement in cash-in-advance models that the nominal interest rates must be nonnegative.

fluctuations.

For the class of New Keynesian models in which the friction is nominal rigidities, the key references are Benigno and Benigno (2003), Corsetti and Pesenti (2005), Devereaux and Sutherland (2008), Corsetti et al. (2008b), and Corsetti et al. (2010). Among these, the focus will be the handbook chapter Corsetti et al. (2010), which builds off the important contributions from Benigno and Benigno (2003) and Corsetti and Pesenti (2005). The setting in this class of models is one of complete markets, or at least a stylized setting in Corsetti and Pesenti (2005) in which the real shocks do not affect the equilibrium consumption and labor choices. Firms operate in a setting of monopolistic competition. With endogenous markups, the degree of exchange rate pass-through can be examined. Nominal rigidities serve as the friction in these settings, where the policy prediction depends upon whether the price rigidity occurs in terms of the producer's currency or in terms of the consumer's currency. Under the former price rigidity (producer currency pricing, or PCP), Corsetti et al. (2010) show that inward rules, or monetary policies that would be optimal in a closed economy (specifically, domestic price stabilization) may not be optimal in an open economy. Optimal policies require policy makers to trade off exchange rate stabilization (to account for the nominal rigidities being imported from foreign firms) with domestic price stabilization (to simultaneously minimize the domestic output gap and the effect of domestic nominal distortions).

The two trade-offs mentioned (domestic price stability and exchange rate stability) do not tell the whole story, as we need to account for a third trade-off arising from financial imperfections (or incomplete markets). Such imperfections are not considered in Corsetti et al. (2010), but are the main focus of this present paper. From the concluding remarks of Corsetti et al. (2010):

> "Key lessons for monetary policy analysis can be learnt from models in which asset markets do not support the efficient allocation" and the "design of monetary policy in models with explicit financial distortions [serve] as a complement."
>
> -Corsetti et al. (2010), pg. 928

For the class of models with incomplete markets, Corsetti et al. (2008b) adds nominal rigidities to the exercise carried out in the incomplete markets setting of Corsetti et al. (2008a) and Devereaux and Sutherland (2008) show that price stability is the optimal policy to mitigate the effects of financial shocks to the interest rate rule. To capture portfolio effects, Devereaux and Sutherland (2008) perform a 2nd order approximation and consider an asset structure (labeled NB- nominal bonds) that is identical to that in the present paper.

Without any real shocks, however, zero inflation monetary policy removes the friction caused by the nominal rigidities, while simultaneously nullifying the interest rate shocks.

Unlike the cited models in the New Keynesian tradition, the important contribution of the present paper is to consider real shocks in which there is an actual trade-off between the welfare effects of inflation (or price changes) and the endogenous asset structure. The paper demonstrates that policies of domestic price stabilization or exchange rate stabilization (or both) are the worst policies for risk-sharing as they reduce the asset structure to a single linearly independent asset.

The related subfield of international finance (see Gourinchas and Rey (2007)) does not explicitly model policy choice, but does stress that shortfalls in current account balances can be remedied with exchange rate movements. The natural connection to the present paper is that optimal monetary policy must operate through movements in the exchange rate.

3 The Model

The model describes a large open economy with N countries $i \in \mathbf{I} = \{1, ..., N\}$, each containing a monetary authority.

Time is discrete and infinite with time periods $t \in \{0, 1, ...\}$. The filtration of uncertainty follows a one-period Markov process with finite state space $\mathbf{S} = \{1, ..., S\}$. The realized state of uncertainty in any period t, denoted s_t, is a function only of the realized state in the previous period $t-1$, denoted s_{t-1}. This random process is characterized by a transition matrix $\Pi \in \mathbb{R}^{S,S}$ whose elements are $\pi(s, s')$ for row s and column s'. This paper focuses on incomplete markets and since there will be N assets, the number of states must be strictly larger than the number of assets.

Assumption 1 $S > N$.

The history of all realizations up to and including the current realization completely characterizes the date-event and is required to uniquely identify the markets, household decisions, and policy choices. Define the history of realizations up to and include the realization s_t in period t as $s^t = (s_0, s_1, ..., s_t)$. For convenience, $\pi(s^\tau | s^t)$ for any $\tau > t$ refers to the probability that history s^τ is realized conditional on the history s^t. Additionally, let $s^{t+j} \succ s^t$ refer to the S^j histories $(s^t, \sigma_1, ..., \sigma_j)_{(\sigma_1, ..., \sigma_j) \in \mathbf{S}^j}$ that are realized j periods from the date-event s^t. The trivial specification $s^{t+0} \succ s^t$ refers to the singleton $\{s^t\}$.

3.1 Households

In each country, a unit mass of homogeneous households live. Households in country $i \in \mathbf{I}$ receive the endowments $e_i = (e_i(s^t))_{s^t \succ s_0; t \geq 0} \in \ell^\infty_{++}$.[5] The endowments are only in terms of the domestic commodity. Households receive zero endowments of any foreign commodities. I assume that the endowments are stationary. Define the stationary endowment mappings as $\mathbf{e}_i : \mathbf{S} \to \mathbb{R}_{++}$ such that $e_i(s^t) = \mathbf{e}_i(s_t)$ for all date-events and all countries $i \in \mathbf{I}$. Denote the aggregate endowment as E: $\mathbf{S} \to \mathbb{R}_{++}$ such that $\mathbf{E}(s) = \sum_{i \in \mathbf{I}} \mathbf{e}_i(s)$ $\forall s \in \mathbf{S}$. The model permits aggregate risk, i.e., $\mathbf{E}(s) \neq \mathbf{E}(\sigma)$ for some $s, \sigma \in \mathbf{S}$.

Households in country i consume commodities sold in all countries. The consumption of country j commodities by the country i household in date-event s^t is denoted $c_{i,j}(s^t) \in \mathbb{R}_+$. The commodities are assumed to be perfect substitutes, meaning that the households only care about the total consumption $c_i(s^t) = \sum_{j \in \mathbf{I}} c_{i,j}(s^t)$.

The vector of consumption for households in country i is denoted $c_i = (c_i(s^t))_{s^t \succ s_0; t \geq 0} \in \ell^\infty_+$.[6]

The household preferences are assumed to be identical and satisfy constant relative risk aversion:

$$\sum_{t=0}^{\infty} \beta^t \sum_{s^t \succ s_0} \pi\left(s^t | s_0\right) u\left(c\left(s^t\right)\right). \tag{1}$$

Assumption 2 The discount factor $\beta \in (0,1)$ and $u(c) = \frac{c^{1-\rho}}{1-\rho}$ for $\rho \geq 0$ and $\rho \neq 1$ and $u(c) = \ln(c)$ for $\rho = 1$.

In each date-event s^t, the money supply in country $i \in \mathbf{I}$ is $M_i(s^t) > 0$ and the nominal price level is $p_i(s^t) > 0$. Let $\xi_i(s^t)$ be the nominal exchange rate for country i, relative to country 1. Specifically, $\xi_i(s^t)$ is the number of units of country 1 currency for each one unit of country i currency. As the numeraire country, $\xi_1(s^t) = 1$.

Each country issues a short-term (one-period) nominally risk-free bond (government debt). The nominal payout of a 1-period bond issued by country i in date-event s^t equals 1 unit of country i currency for all date-events $s^{t+1} \succ s^t$ and 0 otherwise. The nominal asset price for a country $i \in \mathbf{I}$ bond in date-event s^t (in units of the country i currency) is denoted $q_i(s^t)$.

Each date-event is divided into two subperiods. In the initial subperiod, the money markets and bond markets open. Denote $(\hat{m}_{i,j}(s^t))_{j \in \mathbf{I}}$ as the money holdings by country i

[5] ℓ^∞_{++} is the space of bounded and strictly positive infinite sequences under the sup norm.
[6] ℓ^∞_{+} is the space of bounded and nonnegative infinite sequences under the sup norm.

households by the close of the money markets in date-event s^t (in terms of each currency $j \in \mathbf{I}$). By definition, the money holdings are nonnegative. The entire vector of money holdings is denoted $\hat{m}_i(s^t) = (\hat{m}_{i,j}(s^t))_{j \in \mathbf{I}} \in \mathbb{R}_+^N$. Denote $(b_{i,j}(s^t))_{j \in \mathbf{I}}$ as the bond holdings by country i households by the close of the bond markets in date-event s^t. Each bond can either be held long or short by the household. Denote the entire portfolio as $b_i(s^t) = (b_{i,j}(s^t))_{j \in \mathbf{I}} \in \mathbb{R}^N$.

Denote $\omega_i(s^t) \in \mathbb{R}$ as the nominal wealth held by the country i households for use in the date-event s^t. The wealth is specified in units of the country i currency. The initial period value $\omega_i(s_0)$ is a parameter of the model. The budget constraint, at the close of the money markets and bond markets in date-event s^t, is given by:

$$\frac{1}{\xi_i(s^t)} \left({}_{j \in \mathbf{I}} \quad \xi_j \;\; s^t \;\; \hat{m}_{i,j} \;\; s^t \; + q_j \;\; s^t \;\; b_{i,j} \;\; s^t \right) \leq \omega_i \;\; s^t \; . \tag{2}$$

The budget constraint is specified in units of the country i currency.

In the second subperiod of each date-event, the commodity markets opens. The purchase of the commodity is subject to the cash-in-advance constraints:

$$p_j \;\; s^t \;\; c_{i,j} \;\; s^t \; \leq \hat{m}_{i,j} \;\; s^t \quad \forall (i,j) \in \mathbf{I}^2. \tag{3}$$

At the same time that consumption is being purchased on the commodity markets, the households receive income from selling their endowments. Recall that households only receive endowments in the domestic commodities. Denote $m_i(s^t) = (m_{i,j}(s^t))_{j \in \mathbf{I}} \in \mathbb{R}_+^N$ as the money holding of the country i households by the close of the commodity markets in date-event s^t, where

$$\begin{aligned} m_{i,j} \;\; s^t \;\; &= \;\; \hat{m}_{i,j} \;\; s^t \; + p_i \;\; s^t \;\; \mathbf{e}_i(s_t) - p_j \;\; s^t \;\; c_{i,j} \;\; s^t \quad \text{for } i = j. \\ m_{i,j} \;\; s^t \;\; &= \;\; \hat{m}_{i,j} \;\; s^t \; - p_j \;\; s^t \;\; c_{i,j} \;\; s^t \quad \text{for } i = j. \end{aligned} \tag{4}$$

The sequence of money variables are $m_i = (m_i(s^t))_{s^t \succ s_0; t \geq 0}$.

Given the money definition (4), the cash-in-advance constraints (3) can be rewritten as:

$$\begin{aligned} m_{i,j} \;\; s^t \;\; &\geq \;\; p_i \;\; s^t \;\; \mathbf{e}_i(s_t) \; \text{ for } i = j. \\ m_{i,j} \;\; s^t \;\; &\geq \;\; 0 \text{ for } i = j. \end{aligned} \tag{5}$$

Entering into the date-events $s^{t+1} \succ s^t$, the nominal wealth available to country i households (in terms of the country i currency) is equal to the money holding plus the portfolio payout:

$$\omega_i\left(s^{t+1}\right) = \frac{1}{\xi_i\left(s^{t+1}\right)} \sum_{j\in\mathbf{I}} \xi_j\left(s^{t+1}\right)\left(\hat{m}_{i,j}\left(s^t\right) + b_{i,j}\left(s^t\right)\right). \tag{6}$$

Households are permitted to short-sell the nominal bonds, so households in country i must satisfy the following implicit debt constraint for all bonds (in real terms):

$$\inf_{t,s^t}\left(\frac{q_j\left(s^t\right) b_{i,j}\left(s^t\right)}{p_j\left(s^t\right)}\right) > -\infty \text{ for } j \in \mathbf{I}. \tag{7}$$

The debt constraint states that the real debt position must be bounded for all random state realizations. Typically, the implicit debt constraint is specified only in terms of the value of the entire portfolio. Here, however, since the payout matrix is endogenously determined without any guarantees that it has full rank, I need to impose a stronger debt constraint to ensure that the household choice set is compact and an equilibrium exists.

The household optimization problem for households in country i is given by:

$$\begin{array}{ll} \underset{(c_i,b_i,m_i)}{max} & \sum_{t=0}^{\infty} \beta^t \pi\left(s^t|s_0\right) u\left(c_i\left(s^t\right)\right) \\ subj.\ to & \text{budget constraint (2) with (4) and (6) } \forall t, s^t \\ & \text{cash-in-advance constraint (5) } \forall t, s^t \\ & \text{debt constraint (7)} \end{array} \tag{8}$$

3.2 Monetary authorities

The country i monetary authority chooses the debt positions $B_i\left(s^t\right) = \left(B_{i,j}\left(s^t\right)\right)_{j\in\mathbf{I}} \in \mathbb{R}^N$ in each date-event s^t, where $B_{i,i}\left(s^t\right) \geq 0$ is the amount of domestic debt issued and must be nonnegative (foreign debt can be either positive or negative).[7]

The country i monetary authority issues the money supply $M_i\left(s^t\right)$ in the date-event s^t. In the initial period s_0, the country i monetary authority has the nominal obligation $W_i\left(s_0\right)$.

The country i monetary authority has the following budget constraints, where the liabilities of the monetary authority are on the left-hand side of the equations and the assets of

[7] In fact, the monetary authority does not issue debt, but buys or sells the debt issued by the fiscal authority. The fiscal authority's only role in this model is the debt choice, so for simplicity I allow the monetary authority to make this choice directly.

the monetary authority are on the right-hand side of the equations:

$$
\begin{aligned}
W_i(s_0) &= M_i(s_0) + \frac{\sum_{j\in\mathbf{I}} \xi_j(s_0)\, q_j(s_0)\, B_{i,j}(s_0)}{\xi_i(s_0)}. \qquad (9)\\
M_i\left(s^{t-1}\right) + \frac{\sum_{j\in\mathbf{I}} \xi_j(s^t)\, q_j(s^t)\, B_{i,j}(s^{t-1})}{\xi_i(s^t)} &= M_i\left(s^t\right) + \frac{\sum_{j\in\mathbf{I}} \xi_j(s^t)\, q_j(s^t)\, B_{i,j}(s^t)}{\xi_i(s^t)}.
\end{aligned}
$$

As with households, implicit debt constraints are required:

$$
\inf_{t,s^t} \frac{q_j(s^t)\, B_{i,j}(s^t)}{p_j(s^t)} > -\infty \text{ for } j \in \mathbf{I}. \qquad (10)
$$

3.3 Sequential competitive equilibrium

The following equilibrium concept is the most general. It only requires that the monetary authority variables are feasible. It does not require that the monetary authority variables are optimal.

Definition 1 *A sequential competitive equilibrium (SCE) is the household variables* $(c_i, b_i, m_i)_{i\in\mathbf{I}}$, *the monetary authority variables* $\{B_i(s^t), M_i(s^t)\}_{i\in\mathbf{I}}$, *and the price variables* $\{p_i(s^t), \xi_i(s^t), q_i(s^t)\}_{i\in\mathbf{I}}$ *such that:*

1. *Given* $\{p_i(s^t), \xi_i(s^t), q_i(s^t)\}_{i\in\mathbf{I}}$ *and* $\omega_i(s_0)$, *households in country* i *choose the sequence of variables* (c_i, b_i, m_i) *to solve the household problem (8).*

2. *Given* $W_i(s_0)$, *the monetary authority in country* i *chooses the variables* $\{B_i(s^t), M_i(s^t)\}$ *to satisfy (9) and (10).*

3. *Markets clear:*

   (a) $\sum_{i\in\mathbf{I}} c_i(s^t) = \mathbf{E}(s_t)$ *for every* t, s^t.

   (b) $\sum_{i\in\mathbf{I}} \xi_i(s_0)\, \omega_i(s_0) = \sum_{i\in\mathbf{I}} \xi_i(s_0)\, W_i(s_0)$.

   (c) $\sum_{i\in\mathbf{I}} m_{i,j}(s^t) = M_j(s^t)$ *for every* $j \in \mathbf{I}$ *and for every* t, s^t.

   (d) $\sum_{i\in\mathbf{I}} b_{i,j}(s^t) = \sum_{i\in\mathbf{I}} B_{i,j}(s^t)$ *for every* $j \in \mathbf{I}$ *and for every* t, s^t.

Standard existence results ensure that a SCE always exists.

Notice that condition (b) does not specify that the initial nominal obligations of the monetary authorities are owed to a particular set of households.

Given the assumption that the commodities from both countries are perfect substitutes, then the equilibrium exchange rates are given by:

$$\xi_i \left(s^t\right) = \frac{p_1\left(s^t\right)}{p_i\left(s^t\right)} \quad \forall i \in \mathbf{I}.$$

3.4 Quantity Theory of Money

The equilibrium asset prices $q_i\left(s^t\right) \leq 1 \ \forall i \in \mathbf{I}$. Otherwise, the market clearing condition on the bond markets is not satisfied as households prefer to save using money holdings and not bond holdings. If $q_i\left(s^t\right) < 1$, the cash-in-advance constraints (5) associated with country i currency will bind for all households. With binding cash-in-advance constraints (5), the market clearing condition for the money markets implies that the Quantity Theory of Money holds:

$$M_i\left(s^t\right) = p_i\left(s^t\right) \mathbf{e}_i\left(s_t\right). \tag{11}$$

3.5 Friedman rule

The Friedman rule for country i in date-event s^t is such that $q_i\left(s^t\right) = 1$. Under the Friedman rule, money and the 1-period bond are perfect substitutes. Market clearing for both implies that the sum of the two is pinned down for all households and the monetary authorities, but not the composition. The cash-in-advance constraints (5) need not bind under the Friedman rule.

However, it is innocuous (i.e., without real effects) under the Friedman rule to set the household money holdings such that the cash-in-advance constraints (5) bind. This would allow the Quantity Theory of Money (11) to hold.

Suppose the Friedman rule is imposed by all countries in all date-events. The Friedman rule is a special case of inflation rate targeting, where the nominal interest rate is set equal to 0. The remaining sections show that inflation rate targeting policy is consistent with Pareto efficiency, and social welfare is higher under a different form of monetary policy.[8]

[8] If the objective of the monetary authority includes both the maximization of a social welfare function and the minimization of the social cost of holding money (the social cost of holding money is minimized under the Friedman rule), then the optimal policy will involve lower interest rates than predicted in this paper. However, so long as the monetary authority places strictly positive weight on maximizing a social welfare function, the Friedman rule will continue to be a suboptimal policy rule.

3.6 Fiscal Theory of the Price Level

Usually, the initial price levels $(p_i(s_0))_{i\in\mathbf{I}}$ are not pinned down in equilibrium.[9] In this model, however, any equilibrium with a Pareto efficient allocation has a determinate vector of initial prices. This does not require fiscal policy in any form, but only monetary policy and a common social welfare function among monetary authorities.

Specifically, any equilibrium with a Pareto efficient allocation requires a stationary wealth distribution for all households (see Theorem 1). The initial period budget constraints then suffice to determine the initial price levels $(p_i(s_0))_{i\in\mathbf{I}}$ in terms of the initial nominal wealth $(\omega_i(s_0))_{i\in\mathbf{I}}$.

4 Markov Equilibrium

4.1 Constraints in real terms

Define the real debt positions for the monetary authorities and the real bond positions for the households as $\hat{B}_{i,j}(s^t) = \frac{B_{i,j}(s^t)}{p_j(s^t)}$ and $\hat{b}_{i,j}(s^t) = \frac{b_{i,j}(s^t)}{p_j(s^t)}$ for all $(i,j)\in\mathbf{I}^2$. The portfolios are denoted $\hat{B}_i(s^t)$ and $\hat{b}_i(s^t)$ for all $i\in\mathbf{I}$. Market clearing in terms of nominal bond positions occurs if and only if market clearing in the real bond positions occurs.

Additionally, define the stochastic price ratios $\nu_i(s^t) = \frac{p_i(s^{t-1})}{p_i(s^t)}$ for $i\in\mathbf{I}$.

4.1.1 Monetary authority constraints

The monetary authority constraints (9) in real terms, after using the Quantity Theory of Money (11), are given by:

$$\nu_i\left(s^t\right)\mathbf{e}_i(s_{t-1}) + \sum_{j\in\mathbf{I}} \nu_j\left(s^t\right)\hat{B}_{i,j}\left(s^{t-1}\right) = \mathbf{e}_i(s_t) + \sum_{j\in\mathbf{I}} q_j\left(s^t\right)\hat{B}_{i,j}\left(s^t\right) \quad \forall i\in\mathbf{I}. \tag{12}$$

There is a natural indeterminacy in the monetary authority constraints as an objective function has not been specified. Specifically, if the monetary authority in country i increased $\hat{B}_{i,i}(s^t)$ and decreased $\hat{B}_{i,j}(s^t)$ without changing the right-hand side of constraint (12), while the monetary authority in country j decreased $\hat{B}_{j,i}(s^t)$ and increased $\hat{B}_{j,j}(s^t)$ without changing the market clearing sums $B_{i,i}(s^t) + B_{j,i}(s^t)$ and $B_{j,j}(s^t) + B_{i,j}(s^t)$, then Walras' Law dictates that constraint (12) would automatically be satisfied for country j.

[9] Notable works on the fiscal theory of the price level include Leeper (1991), Sims (1994), and Woodford (1994, 1995).

The paper adopts the standard convention of viewing policy as the choices of the variables $\{\nu_i(s^t)\}_{i\in\mathbf{I}}$, and defines the inflation rate targeting rules, interest rate targeting rules, and nominal GDP targeting rules in terms of restrictions on $\{\nu_i(s^t)\}_{i\in\mathbf{I}}$. However, from a theoretical point of view, the actual choices of the monetary authorities will be the debt positions $\left\{\hat{B}_i(s^t)\right\}_{i\in\mathbf{I}}$.

For each vector $\left\{\hat{B}_i(s^t)\right\}_{i\in\mathbf{I}}$, there exists a unique vector $\{\nu_i(s^t)\}_{i\in\mathbf{I}}$. Define the $N\times N$ debt position matrix

$$\hat{B}\left(s^t\right) = \left[\hat{B}_1(s^t) \quad ... \quad \hat{B}_N(s^t)\right], \tag{13}$$

where all vectors, including $\hat{B}_i(s^t)$, are considered column vectors unless otherwise specified. The monetary authority constraints (12), for all countries $i\in\mathbf{I}$ in date-event s^t, are written in matrix notation as:

$$\left(\nu_i\left(s^t\right)\right)_{i\in\mathbf{I}}^T\left(diag\left(\mathbf{e}_i(s_{t-1})\right)_{i\in\mathbf{I}}+\hat{B}\left(s^{t-1}\right)\right)=\left(\mathbf{e}_i(s_t)\right)_{i\in\mathbf{I}}^T+\left(q_i\left(s^t\right)\right)_{i\in\mathbf{I}}^T\left(\hat{B}\left(s^t\right)\right). \tag{14}$$

Provided that the monetary authorities adopt policies such that $\left[diag\left(\mathbf{e}_i(s_{t-1})\right)_{i\in\mathbf{I}}+\hat{B}(s^{t-1})\right]$ is a full rank matrix, then the formula for $(\nu_i(s^t))_{i\in\mathbf{I}}^T$ is given by:

$$\left(\nu_i\left(s^t\right)\right)_{i\in\mathbf{I}}^T=\left[diag\left(\mathbf{e}_i(s_{t-1})\right)_{i\in\mathbf{I}}+\hat{B}\left(s^{t-1}\right)\right]^{-1}\left[\left(\mathbf{e}_i(s_t)\right)_{i\in\mathbf{I}}^T+\left(q_i\left(s^t\right)\right)_{i\in\mathbf{I}}^T\left(\hat{B}\left(s^t\right)\right)\right]. \tag{15}$$

4.1.2 Household problem

The household problem will be recursive in terms of wealth. If the cash-in-advance constraint (5) is binding, the budget constraint for households in country i are given by:

$$c_i\left(s^t\right)+\sum_{j\in\mathbf{I}}q_j\left(s^t\right)\hat{b}_{i,j}\left(s^t\right)\leq\nu_i\left(s^t\right)\mathbf{e}_i(s_{t-1})+\sum_{j\in\mathbf{I}}\nu_j\left(s^t\right)\hat{b}_{i,j}\left(s^{t-1}\right). \tag{16}$$

Define the real wealth for household h entering date-event s^t as

$$\hat{\omega}_i(s^t)=\frac{\omega_i(s^t)}{p_i(s^t)}=\nu_i\left(s^t\right)\mathbf{e}_i(s_{t-1})+\sum_{j\in\mathbf{I}}\nu_j\left(s^t\right)\hat{b}_{i,j}\left(s^{t-1}\right).$$

The first order conditions with respect to the bond holdings are given by:

$$q_j\left(s^t\right)=\beta\sum_{\sigma\in\mathbf{S}}\pi(s_t,\sigma)\left(\frac{c_i(s^t,\sigma)}{c_i(s^t)}\right)^{-\rho}\nu_j\left(s^t,\sigma\right)\quad\forall(i,j)\in\mathbf{I}^2. \tag{17}$$

Notice that the first order conditions hold for the consumers in all countries $i \in \mathbf{I}$. With complete markets, this implies that the ratios of marginal utilities are proportional for all households. However, with incomplete markets, this is no longer true as the ratio of marginal utilities belongs to a linear subspace of dimension $\mathbb{R}^{S-N}_{++}$.

4.2 Markov equilibrium

For simplicity, define $\omega(s^t) = (\omega_i(s^t))_{i\in\mathbf{I}}$, $W(s^t) = (W_i(s^t))_{i\in\mathbf{I}}$, $\hat{\omega}(s^t) = (\hat{\omega}_i(s^t))_{i\in\mathbf{I}}$, $p(s^t) = (p_i(s^t))_{i\in\mathbf{I}}$, $\nu(s^t) = (\nu_i(s^t))_{i\in\mathbf{I}}$, $q(s^t) = (q_i(s^t))_{i\in\mathbf{I}}$, $\xi(s^t) = (\xi_i(s^t))_{i\in\mathbf{I}}$, $c(s^t) = (c_i(s^t))_{i\in\mathbf{I}}$, $\hat{b}(s^t) = \left(\hat{b}_i(s^t)\right)_{i\in\mathbf{I}}$, and $\hat{B}(s^t) = \left(\hat{B}_i(s^t)\right)_{i\in\mathbf{I}}$.

The initial conditions are s_0, $\omega(s_0)$, $W(s_0)$, and $p(s_0)$. The prices $p(s_0)$ are chosen by the monetary authority. A Markov equilibrium is defined in terms of a policy correspondence (a set-valued mapping) and a transition correspondence. These mappings, together with the initial conditions, determine the entire sequence of SCE variables.

In this paper, the policy vector will be the vector of asset prices. The focus of the paper is on policies that support a Pareto efficient allocation. Necessary conditions for Pareto efficiency are that the policy vectors are stationary. Stationary policy vectors also allow for Pareto inefficient equilibrium allocations, so I am not assuming Pareto efficiency by restricting attention to stationary policies. Given this, and the desire to maintain simplicity of the equilibrium concept and the eventual computation of such equilibria, I restrict attention to stationary policy vectors.

This simply means that country $i \in \mathbf{I}$ commits to the state-contingent policy rule $(\mathbf{q}_i(s))_{s\in\mathbf{S}} \in \mathbb{R}^S_+$. The policy rule dictates that $q_i(s^t) = \mathbf{q}_i(s_t)$ for all date-events s^t and $\forall i \in \mathbf{I}$.

It would be equivalent to specify that country $i \in \mathbf{I}$ targets $\left\{\hat{B}_i(s^t)\right\}$ instead of $\{q_i(s^t)\}$. Both vectors are of equal size.

The real wealth distribution is $\hat{\omega}(s^t) \in \mathbb{R}^N$. Notice that (i) the possibility of borrowing in date-event s^t allows for negative wealth values and (ii) the wealth distribution is $N-$dimensional as the total household wealth depends upon the choices of the monetary authorities and does not equal a fixed parameter.

4.2.1 State space

The state space includes the aggregate shock realization in the current period, the wealth distribution, the household portfolios, and the debt positions for all but one country's monetary authority. The state space is $\Omega = \mathbf{S} \times \mathbb{R}^N \times \mathbb{R}^{N^2} \times \mathbb{R}^{N(N-1)}$ with typical element $\left(s, \hat{\omega}(s^t), \hat{b}(s^t), \hat{B}_{-N}(s^t)\right)$.

4.2.2 Expectations correspondence

Define $\hat{\mathbf{Z}} = \mathbb{R}^N_+ \times \mathbb{R}^N \times \mathbb{R}^N_+ \times \mathbb{R}^N_+ \times \mathbb{R}^N_+$ as the set of current period variables, with typical element

$$\hat{z}\left(s^t\right) = \left(c\left(s^t\right), \hat{B}_N\left(s^t\right), \nu\left(s^t\right), M\left(s^t\right), p\left(s^t\right)\right).$$

Define $\mathbf{Z} = \mathbb{R}^N \times \mathbb{R}^{N^2} \times \mathbb{R}^{N(N-1)} \times \hat{\mathbf{Z}}$ as the set containing the state variables $\left(\hat{\omega}(s^t), \hat{b}\left(s^t\right), \hat{B}_{-N}\left(s^t\right)\right)$ and the set of current period variables $\hat{z}\left(s^t\right)$. For simplicity, define

$$z\left(s^t\right) = \left(\hat{\omega}(s^t), \hat{b}\left(s^t\right), \hat{B}_{-N}\left(s^t\right), \hat{z}\left(s^t\right)\right).$$

The key mapping for existence is the expectations correspondence

$$g : \Omega \times \hat{\mathbf{Z}} \rightrightarrows (\mathbf{Z})^S$$

that describes all next period variables that are consistent with the budget constraints, household optimization, and market clearing. The expectations correspondence is defined such that for z and $\left(z'\left(\sigma\right)\right)_{\sigma \in \mathbf{S}}$, where

$$\begin{aligned} z &= \left(\hat{\omega}, \hat{b}, \hat{B}, c, \nu, M, p\right) \\ z'\left(\sigma\right) &= \left(\hat{\omega}'\left(\sigma\right), \hat{b}'\left(\sigma\right), \hat{B}'\left(\sigma\right), c'\left(\sigma\right), \nu'\left(\sigma\right), M'\left(\sigma\right), p'\left(\sigma\right)\right) \quad \forall \sigma \in \mathbf{S}, \end{aligned}$$

the vector of variables $\left(z'\left(1\right), ..., z'\left(S\right)\right) \in g\left(s, z\right)$ if the following conditions hold.

1. For all $\sigma \in \mathbf{S}$, the nominal price levels

$$p'_i\left(\sigma\right) = \frac{p_i(s)}{\nu'_i\left(\sigma\right)} \text{ for } i \in \mathbf{I}. \tag{18}$$

2. For all $\sigma \in \mathbf{S}$, the money supplies

$$M'_i\left(\sigma\right) = p'_i\left(\sigma\right) \mathbf{e}_i\left(\sigma\right) \text{ for } i \in \mathbf{I}. \tag{19}$$

3. For all $\sigma \in \mathbf{S}$, the household wealth

$$\hat{\omega}'_i(\sigma) = \nu'_i\left(\sigma\right) \mathbf{e}_i\left(s\right) + \sum_{j \in \mathbf{I}} \nu'_j\left(\sigma\right) \hat{b}_{i,j} \text{ for } i \in \mathbf{I}. \tag{20}$$

4. For all $\sigma \in \mathbf{S}$, the monetary authority constraints (12) are satisfied:

$$\nu'(\sigma)^T \left(diag\left(\mathbf{e}_i(s)\right)_{i\in\mathbf{I}} + \hat{B} \right) = \left(\mathbf{e}_i(\sigma)\right)^T_{i\in\mathbf{I}} + \mathbf{q}(\sigma)^T \left(\hat{B}'(\sigma) \right), \tag{21}$$

where with slight abuse of notation, the matrices

$$\hat{B} = \left(\hat{B}_1 \quad ... \quad \hat{B}_N \right) \qquad \hat{B}'(\sigma) = \left(\hat{B}'_1(\sigma) \quad ... \quad \hat{B}'_N(\sigma) \right).$$

5. For all $\sigma \in \mathbf{S}$, the household consumptions satisfy the budget constraint:

$$c'_i(\sigma) + \sum_{j\in\mathbf{I}} \mathbf{q}_j(\sigma)\,\hat{b}'_{i,j}(\sigma) = \hat{\omega}'_i(\sigma) \text{ for } i \in \mathbf{I}. \tag{22}$$

6. For all $i, j \in \mathbf{I}^2$, the Euler equation (17):

$$\mathbf{q}_j(s) = \beta \sum_{\sigma\in\mathbf{S}} \pi(s,\sigma) \left(\frac{c'_i(\sigma)}{c_i} \right)^{-\rho} \nu'_j(\sigma). \tag{23}$$

7. For all $j \in \mathbf{I}$ and all $\sigma \in \mathbf{S}$, markets clear:

$$\sum_{i\in\mathbf{I}} \hat{B}'_{i,j}(\sigma) = \sum_{i\in\mathbf{I}} \hat{b}'_{i,j}(\sigma). \tag{24}$$

By definition, the graph of g is a closed subset of $\Omega \times \hat{\mathbf{Z}} \times (\mathbf{Z})^S$.

4.2.3 Markov equilibrium definition

Claim 1 *For all* $\left(\sigma, \hat{\omega}'(\sigma), \hat{b}'(\sigma), \hat{B}'_{-N}(\sigma)\right) \in \Omega$, *there exists a unique vector*

$$\hat{z}'(\sigma) = \left(c'(\sigma), \hat{B}'_N(\sigma), \nu'(\sigma), M'(\sigma), p'(\sigma) \right)$$

satisfying (18), (19), (21), (22), and (24).

Proof. The value $\hat{B}'_N(\sigma)$ is determined from the market clearing conditions (24). The monetary authority constraints (21) imply unique values for $\nu'(\sigma)$. Given $\left(\hat{\omega}'(\sigma), \hat{b}'(\sigma)\right)$, the budget constraint (22) yields the unique vector $c'(\sigma)$. Given $\nu'(\sigma)$, equation (18) yields the unique values for $p'(\sigma)$ and equation (19) yields the unique values for $M'(\sigma)$. ■

Define the function $\phi : \Omega \to \hat{\mathbf{Z}}$ such that

$$\phi\left(\sigma, \hat{\omega}'(\sigma), \hat{b}'(\sigma), \hat{B}'_{-N}(\sigma)\right) = \left\{\hat{z}'(\sigma) : (18), (19), (21), (22), \text{and } (24) \text{ satisfied}\right\}.$$

From Claim 1, ϕ is a well-defined function.

A Markov equilibrium is defined by a policy correspondence $\mathbf{V} : \mathbf{S} \times \mathbb{R}^N \rightrightarrows \mathbb{R}^{N^2} \times \mathbb{R}^{N(N-1)}$ and a transition correspondence $\mathbf{F}_\sigma : graph(\mathbf{V}) \rightrightarrows \mathbb{R}^N \times \mathbb{R}^{N^2} \times \mathbb{R}^{N(N-1)}$ for all $\sigma \in \mathbf{S}$ satisfying the following two properties:

1. For all $\left(s, \hat{\omega}, \hat{b}, \hat{B}_{-N}\right) \in graph(\mathbf{V})$, then for all $\sigma \in \mathbf{S}$,

$$\left(\mathbf{F}_\sigma\left(s, \hat{\omega}, \hat{b}, \hat{B}_{-N}\right), \phi\left(\sigma, \mathbf{F}_\sigma\left(s, \hat{\omega}, \hat{b}, \hat{B}_{-N}\right)\right)\right) \subseteq g\left(\sigma, \hat{\omega}, \hat{b}, \hat{B}_{-N}, \phi\left(s, \hat{\omega}, \hat{b}, \hat{B}_{-N}\right)\right).$$

2. For all $\left(s, \hat{\omega}, \hat{b}, \hat{B}_{-N}\right) \in graph(\mathbf{V})$ and all $\sigma \in \mathbf{S}$,

$$\left(\sigma, \mathbf{F}_\sigma\left(s, \hat{\omega}, \hat{b}, \hat{B}_{-N}\right)\right) \subseteq graph(\mathbf{V}).$$

4.3 Markov equilibrium existence

The existence of a SCE suffices to ensure that $\left(\hat{b}(s^t), \hat{B}_{-N}(s^t)\right)$ lies in a compact set for all date-events. Denote this compact set $\Delta \subseteq \mathbb{R}^{N^2} \times \mathbb{R}^{N(N-1)}$.

The next steps provide an iterative algorithm to determine the policy correspondence $\mathbf{V}$. Define the initial correspondence $\mathbf{V}^0 : \mathbf{S} \times \mathbb{R}^N \rightrightarrows \mathbb{R}^{N^2} \times \mathbb{R}^{N(N-1)}$ such that $\mathbf{V}^0(s, \hat{\omega}) = \Delta$ for all $(s, \hat{\omega}) \in \mathbf{S} \times \mathbb{R}^N$. Define the operator G_Δ that maps from the correspondence $\mathbf{V}^n : \mathbf{S} \times \mathbb{R}^N \rightrightarrows \mathbb{R}^{N^2} \times \mathbb{R}^{N(N-1)}$ to a new correspondence $\mathbf{V}^{n+1} : \mathbf{S} \times \mathbb{R}^N \rightrightarrows \mathbb{R}^{N^2} \times \mathbb{R}^{N(N-1)}$ as follows:

$$\mathbf{V}^{n+1}(s, \hat{\omega}) = \left\{ \left(\hat{b}, \hat{B}_{-N}\right) \in \Delta : \begin{array}{l} \text{(i) } (\hat{\omega}'(\sigma))_{\sigma \in \mathbf{S}} \text{ satisfy (20) given } \left(\hat{b}, \hat{B}_{-N}, (\nu'(\sigma))_{\sigma \in \mathbf{S}}\right) \\ \text{(ii) } \left(\hat{b}, \hat{B}_{-N}, (\nu'(\sigma))_{\sigma \in \mathbf{S}}\right) \text{ satisfy (23) given } (\hat{\omega}'(\sigma))_{\sigma \in \mathbf{S}} \\ \text{and for } \left(\hat{b}'(\sigma), \hat{B}'_{-N}(\sigma)\right) \in \mathbf{V}^n(\sigma, \hat{\omega}'(\sigma)) \end{array} \right\}.$$

In words, given the correspondence $\mathbf{V}^n : \mathbf{S} \times \mathbb{R}^N \rightrightarrows \mathbb{R}^{N^2} \times \mathbb{R}^{N(N-1)}$, the solution to two nonlinear systems of equations determines the image of the new correspondence $\mathbf{V}^{n+1} : \mathbf{S} \times \mathbb{R}^N \rightrightarrows \mathbb{R}^{N^2} \times \mathbb{R}^{N(N-1)}$.

The first nonlinear system of equations uses $\left(\hat{b}, \hat{B}_{-N}, (\nu'(\sigma))_{\sigma \in \mathbf{S}}\right)$ to determine $(\hat{\omega}'(\sigma))_{\sigma \in \mathbf{S}}$ using equation (20).

The second nonlinear system of equations takes as given $(\hat{\omega}'(\sigma))_{\sigma \in \mathbf{S}}$, meaning that the vector $\left(\hat{b}'(\sigma), \hat{B}'_{-N}(\sigma)\right) \in \mathbf{V}^n(\sigma, \hat{\omega}'(\sigma))$ is taken as given as well, for all $\sigma \in \mathbf{S}$. Market clearing (24) uniquely determines $\hat{B}'_N(\sigma)$. Given $\hat{\omega}'_i(\sigma)$ and $\hat{b}'_i(\sigma)$, budget constraint (22) determines a unique value for $c'_i(\sigma)$. With $\hat{\omega}_i$, the budget constraint in the current period implies that the consumption choice c_i is a function of $\hat{b}_i$:

$$c_i = \hat{\omega}_i - \sum_{j \in \mathbf{I}} \mathbf{q}_j\left(\sigma\right) \hat{b}_{i,j}.$$

The values for $\hat{b}$ determine the values for $\hat{B}$ and the values for $\left(\nu'\left(\sigma\right)\right)_{\sigma \in \mathbf{S}}$ (from (21)). Thus, the values for $\hat{b}$ must be chosen to satisfy the Euler equations (23).

Define $\mathbf{V}^* : \mathbf{S} \times \mathbb{R}^N \rightrightarrows \mathbb{R}^{N^2} \times \mathbb{R}^{N(N-1)}$ such that

$$\mathbf{V}^*\left(s, \hat{\omega}\right) = \bigcap_{n=0}^{\infty} \mathbf{V}^n\left(s, \hat{\omega}\right) \quad \text{for all} \quad \left(s, \hat{\omega}\right) \in \mathbf{S} \times \mathbb{R}^N.$$

Claim 2 *A Markov equilibrium exists.*

Proof. Using the iterative algorithm above, Theorem 1 of Kubler and Schmedders (2003) proves that $\mathbf{V}^* : \mathbf{S} \times \mathbb{R}^N \rightrightarrows \mathbb{R}^{N^2} \times \mathbb{R}^{N(N-1)}$ is nonempty valued. Therefore, a Markov equilibrium with policy correspondence $\mathbf{V}^*$ exists. ■

A Markov equilibrium is, by construction, a SCE. There may certainly exist SCE that are not consistent with the recursive form of a Markov equilibrium, but since the policy vectors are stationary, then all SCE will also be Markov equilibria.

5 Necessary Conditions for Pareto Efficiency

5.1 Equations under Pareto efficiency

Given that preferences are identical and homothetic, a Pareto efficient allocation must be such that for some values $\left(\theta_i\right)_{i \in \mathbf{I}} \in \Delta^{N-1} : c_i\left(s^t\right) = \theta_i \mathbf{E}\left(s_t\right)$ for all $i \in \mathbf{I}$ and for all date-events.

From (20) and (22), the budget constraints for the households at the Pareto efficient allocation are given by:

$$\theta_i \mathbf{E}\left(\sigma\right) + \sum_{j \in \mathbf{I}} \mathbf{q}_j\left(\sigma\right) \hat{b}'_{i,j}\left(s^t, \sigma\right) = \hat{\omega}'_i(s^t, \sigma). \tag{25}$$

Theorem 1 *To support a Pareto efficient equilibrium allocation (which is stationary), the wealth distribution $\hat{\omega}'(s^t, \sigma)$ must be a stationary distribution, in which the household wealth only depends upon the current shock σ.*

Proof. See Section A.1. ■

The wealth vectors are defined as:

$$\hat{\omega}_i'(s^t,\sigma) = \nu_i'\left(s^t,\sigma\right)\mathbf{e}_i(s) + \sum_{j\in\mathbf{I}} \nu_j'\left(s^t,\sigma\right)\hat{b}_{i,j}\left(s^t\right). \tag{26}$$

The Euler equations associated with a Pareto efficient allocation are:

$$\mathbf{q}_j(s) = \beta \sum_{\sigma\in\mathbf{S}} \pi(s,\sigma)\left(\frac{\mathbf{E}(\sigma)}{\mathbf{E}(s)}\right)^{-\rho} \nu_j'\left(s^{t-1},s,\sigma\right) \quad \text{for } j\in\mathbf{I}. \tag{27}$$

Given that the asset prices are stationary, (27) implies that $\nu'(s^{t-1},s,\sigma)$ only depends upon (s,σ). Given that the wealth vectors are stationary, (26) implies that the bond holdings $\hat{b}(s^t)$ only depend on s_t. Define $\left(\boldsymbol{\nu}_j'(s,\sigma)\right)_{(j,s,\sigma)\in\mathbf{I}\times\mathbf{S}^2}$ such that $\nu_j'(s^{t-1},s,\sigma) = \boldsymbol{\nu}_j'(s,\sigma)$ for $j\in\mathbf{I}$ and all date-events. Define $\left(\hat{\mathbf{b}}_{i,j}(s)\right)_{(i,j,s)\in\mathbf{I}^2\times\mathbf{S}}$ such that $\hat{b}_{i,j}(s^t) = \hat{\mathbf{b}}_{i,j}(s_t)$ for all $(i,j)\in\mathbf{I}^2$ and all date-events.

In similar fashion, the debt positions for the monetary authorities must also be stationary, so define $\left(\hat{\mathbf{B}}_{i,j}(s)\right)_{(i,j,s)\in\mathbf{I}^2\times\mathbf{S}}$ such that $\hat{B}_{i,j}(s^t) = \hat{\mathbf{B}}_{i,j}(s_t)$ for all $(i,j)\in\mathbf{I}^2$ and all date-events.

5.2 Inflation rate targeting

Suppose that all countries adopt inflation rate targeting rules, but not specific values for the targets. This implies that $\boldsymbol{\nu}_i'(s,\sigma) = \rho_i \;\forall (i,s,\sigma)\in\mathbf{I}\times\mathbf{S}^2$ for the target variables $(\rho_i)_{i\in\mathbf{I}}$. From (27), the asset prices are given by:

$$\mathbf{q}_i(s) = \beta\rho_i \sum_{\sigma\in\mathbf{S}} \pi(s,\sigma)\left(\frac{\mathbf{E}(\sigma)}{\mathbf{E}(s)}\right)^{-\rho} \quad \forall (i,s)\in\mathbf{I}\times\mathbf{S}.$$

This restriction on the stochastic price ratios implies that the bond holdings are constant:

$$\begin{aligned}
\mathbf{b}_{i,j} &= \mathbf{e}_i(s) + \hat{\mathbf{b}}_{i,j}(s) \;\; \forall s\in\mathbf{S}, \text{ when } i=j. \\
\mathbf{b}_{i,j} &= \hat{\mathbf{b}}_{i,j}(s) \;\; \forall s\in\mathbf{S}, \text{ when } i=j.
\end{aligned}$$

In similar fashion, the debt positions are constant:

$$\begin{aligned}
\mathbf{B}_{i,j} &= \mathbf{e}_i(s) + \hat{\mathbf{B}}_{i,j}(s) \;\; \forall s\in\mathbf{S}, \text{ when } i=j. \\
\mathbf{B}_{i,j} &= \hat{\mathbf{B}}_{i,j}(s) \;\; \forall s\in\mathbf{S}, \text{ when } i=j.
\end{aligned}$$

From (20) and (25), the SN household budget constraints are given by:

$$\theta_i \mathbf{E}(s) - \mathbf{q}_i(s)\,\mathbf{e}_i(s) + \sum_{j\in\mathbf{I}} \mathbf{q}_j(s)\,\mathbf{b}_{i,j} = \sum_{j\in\mathbf{I}} \rho_j \mathbf{b}_{i,j} \quad \forall\,(i,s) \in \mathbf{I}\times\mathbf{S}. \tag{28}$$

The $S(N-1)$ monetary authority constraints (12) for countries 1 through $N-1$ are given by:

$$\mathbf{e}_i(s) - \mathbf{q}_i(s)\,\mathbf{e}_i(s) + \sum_{j\in\mathbf{I}} \mathbf{q}_j(s)\,\mathbf{B}_{i,j} = \sum_{j\in\mathbf{I}} \rho_j \mathbf{B}_{i,j} \quad \forall\,(i,s) \in \mathbf{I}\backslash\{N\}\times\mathbf{S}. \tag{29}$$

Walras' Law dictates that the monetary authority constraints for country N are trivially satisfied given market clearing and the above constraints. The total number of equations is $S\,(2N-1)$.

The equilibrium variables are given in the following table:

inflation rate targets	$(\rho_i)_{i\in\mathbf{I}}$	N variables
stationary debt positions	$(\mathbf{B}_i)_{i\neq N}$	$N(N-1)$ variables
stationary bond holdings	$\mathbf{b}$	N^2 variables
allocation distribution	$(\theta_i)_{i\in\mathbf{I}}$	$N-1$ variable

Table 1: Equilibrium variables

Notice that market clearing trivially yields the stationary debt positions $\mathbf{B}_N$.

The payout matrix is given by:

$$[\nu'] = \begin{bmatrix} \boldsymbol{\nu}'_1(s,1) & .. & \boldsymbol{\nu}'_N(s,1) \\ \vdots & .. & \vdots \\ \boldsymbol{\nu}'_1(s,S) & .. & \boldsymbol{\nu}'_N(s,S) \end{bmatrix} = \begin{bmatrix} \rho_1 & .. & \rho_N \\ \vdots & .. & \vdots \\ \rho_1 & .. & \rho_N \end{bmatrix}.$$

Denote $R = rank\ [\nu']$. Under inflation rate targeting, $R = 1$. The Euler equation (27) implies that the asset price vectors are all proportional: $(\mathbf{q}_i(s))_{s\in\mathbf{S}} \propto (\mathbf{q}_1(s))_{s\in\mathbf{S}}$ for all $i \in \mathbf{I}$. This implies that all but one of the bonds is redundant as there only exists 1 linearly independent asset among them.

Theorem 2 *If $S \leq 2$ (more variables than equations), then generically over the subset of household endowments $(\mathbf{e}_i(s))_{(i,s)\in\mathbf{I}\times\mathbf{S}}$, the $S\,(2N-1)$ equations are linearly independent.*

Proof. See Section A.2. ■

Theorem 2 dictates that $S \leq 2$ is a necessary condition (generically) for a Pareto efficient equilibrium allocation. Assumption 1 requires that $S > N$, meaning that it is not possible (generically) for a Pareto efficient allocation to be supported by equilibrium policies in which all countries adopt inflation rate targeting.

5.3 Interest rate targeting

Suppose that all countries adopt interest rate targeting, but do not specify target values. This means $\mathbf{q}_i(s) = \kappa_i \ \ \forall (i,s) \in \mathbf{I} \times \mathbf{S}$ for the target variables $(\kappa_i)_{i \in \mathbf{I}}$.

The Euler equation (27) can be inverted to yield:

$$(\boldsymbol{\nu}'_i(\sigma))_{\sigma \in \mathbf{S}} = \ \beta\hat{\Pi} \ ^{-1} \ (\mathbf{q}_i(\sigma))_{\sigma \in \mathbf{S}} \quad \text{for } i \in \mathbf{I}, \tag{30}$$

where $\hat{\Pi}(s,\sigma) = \pi(s,\sigma) \ \ \frac{\mathbf{E}(\sigma)}{\mathbf{E}(s)} \ ^{-\rho}$. Given the interest rate targeting, then:

$$(\boldsymbol{\nu}'_i(\sigma))_{\sigma \in \mathbf{S}} = \kappa_i \ \ \beta\hat{\Pi} \ ^{-1} \overrightarrow{1} \quad \text{for } i \in \mathbf{I}. \tag{31}$$

Notice that without aggregate risk, inflation rate targeting and interest rate targeting are identical as both imply $(\boldsymbol{\nu}'_i(\sigma))_{\sigma \in \mathbf{S}} \propto \overrightarrow{1}$ and $(\mathbf{q}_i(\sigma))_{\sigma \in \mathbf{S}} \propto \overrightarrow{1}$ for $i \in \mathbf{I}$ (recall the classical Fisher equation).

The fixed bond holdings are defined as previously shown. Additionally, the same $S(2N-1)$ equations must be satisfied.

The payout matrix is given by:

$$[\nu'] = \begin{bmatrix} \boldsymbol{\nu}'_1(s,1) & .. & \boldsymbol{\nu}'_N(s,1) \\ : & .. & : \\ \boldsymbol{\nu}'_1(s,S) & .. & \boldsymbol{\nu}'_N(s,S) \end{bmatrix} = \ \kappa_1 \ \ \beta\hat{\Pi} \ ^{-1} \overrightarrow{1} \ \ .. \ \ \kappa_N \ \ \beta\hat{\Pi} \ ^{-1} \overrightarrow{1} \ \ .$$

The $rank\,[\nu'] = 1$, same as under inflation rate targeting. Theorem 2 continues to hold. The necessary condition is $S \leq 2$, meaning it is not possible (generically) for a Pareto efficient allocation to be supported by equilibrium policies in which all countries adopt interest rate targeting.

Exchange rate stabilization refers to policies such that $\xi_j(s^t,\sigma) = \xi_j(s^t)$ for all countries $i \in \mathbf{I}$ and all realizations $\sigma \in \mathbf{S}$. The equation $\xi_j(s^t,\sigma) = \xi_j(s^t)$ implies $\boldsymbol{\nu}'_j(s,\sigma) = \boldsymbol{\nu}'_1(s,\sigma)$ $\forall (j,s,\sigma) \in \mathbf{I} \times \mathbf{S}^2$. This implies that $[\nu']$ has rank 1, same as under inflation rate targeting and interest rate targeting.

5.4 Nominal GDP targeting

Suppose that all countries adopt nominal GDP targeting without selecting a specific target. This means $\boldsymbol{\nu}'_i(s,\sigma) = \mu_i \frac{\mathbf{e}_i(\sigma)}{\mathbf{e}_i(s)} \ \forall (i,s,\sigma) \in \mathbf{I} \times \mathbf{S}^2$ for the target variables $(\mu_i)_{i \in \mathbf{I}}$. Nominal GDP targeting requires a constant growth rate for the money supply, which is equivalent to the specified policy vector after applying the Quantity Theory of Money (11). From the Euler equation (27): $\mathbf{q}_i(s) = \beta\mu_i \sum_{\sigma\in\mathbf{S}} \pi(s,\sigma) \left(\frac{\mathbf{E}(\sigma)}{\mathbf{E}(s)}\right)^{-\rho} \frac{\mathbf{e}_i(\sigma)}{\mathbf{e}_i(s)}\ \forall(i,s)\in\mathbf{I}\times\mathbf{S}$.

The bond holdings are still fixed (in order for the wealth distribution to be stationary), but the fixed holdings are given by:

$$\mathbf{b}_{i,j} = \frac{\hat{\mathbf{b}}_{i,j}(s)}{\mathbf{e}_j(s)}\ \ \forall(i,j,s)\in\mathbf{I}^2\times\mathbf{S}.$$

In similar fashion, the debt positions can be defined as:

$$\mathbf{B}_{i,j} = \frac{\hat{\mathbf{B}}_{i,j}(s)}{\mathbf{e}_j(s)}\ \ \forall(i,j,s)\in\mathbf{I}^2\times\mathbf{S}.$$

It is straightforward for any country $i\in\mathbf{I}$ to adopt a nominal GDP target $\mu_i = 1$. This is accomplished by setting $\mathbf{B}_i = \left(\hat{\mathbf{B}}_i(s)\right)_{s\in\mathbf{S}} = 0$. This trivially implies that $\boldsymbol{\nu}'_i(s,\sigma) = \frac{\mathbf{e}_i(\sigma)}{\mathbf{e}_i(s)}$ from (21) for all date-events.

Define $\mathbf{I}^* = \{1,...,N^*\}$ as the set of countries that adopt a different nominal GDP target $\mu_i = 1$. For the remaining countries, the target $(\mu_i)_{i\in\mathbf{I}\setminus\mathbf{I}^*} = \overrightarrow{1}$ is fixed, the debt positions $(\mathbf{B}_i)_{i\in\mathbf{I}\setminus\mathbf{I}^*}$ are trivially fixed (at 0), and the monetary authority constraints (21) are trivially satisfied. From (20) and (25), the $S(N-1)$ household budget constraints are given by:

$$\theta_i\mathbf{E}(s) + \sum_{j\in\mathbf{I}} \mathbf{q}_j(s)\,\mathbf{e}_j(s)\mathbf{b}_{i,j} = \mu_i\mathbf{e}_i(s) + \sum_{j\in\mathbf{I}} \mu_j\mathbf{e}_j(s)\mathbf{b}_{i,j}\ \ \forall(i,s)\in\mathbf{I}\setminus\{N\}\times\mathbf{S}. \tag{32}$$

The SN^* monetary authority constraints (12) for countries $i\in\mathbf{I}^*$ are given by:

$$\mathbf{e}_i(s) + \sum_{j\in\mathbf{I}} \mathbf{q}_j(s)\,\mathbf{e}_j(s)\mathbf{B}_{i,j} = \mu_i\mathbf{e}_i(s) + \sum_{j\in\mathbf{I}} \mu_j\mathbf{e}_j(s)\mathbf{B}_{i,j}\ \ \forall(i,s)\in\mathbf{I}\setminus\{N\}\times\mathbf{S}. \tag{33}$$

The total number of equations remaining is $S(N+N^*-1)$.

The payout matrix $[\nu']$ depends upon the current state s, but has the same rank and column space as the endowment matrix

$$E = \begin{bmatrix} \mathbf{e}_1(1) & .. & \mathbf{e}_N(1) \\ \vdots & .. & \vdots \\ \mathbf{e}_1(S) & .. & \mathbf{e}_N(S) \end{bmatrix}.$$

The payout matrix $[\nu']$ could have rank equal to N, in sharp contrast to the result with both inflation rate targeting and interest rate targeting. Denote $R = rank\ [\nu']$, where $R\leq N$. If $R<N$, then the $(N-R)$ redundant variables (for both households and monetary authorities)

must be removed, reducing the number of independent variables. The equilibrium variables are given below:

inflation rate targets	$(\mu_i)_{i\in\mathbf{I}^*}$	N^* variables
stationary debt positions	$\mathbf{B}$	RN^* variables
stationary bond holdings	$(\mathbf{b}_i)_{i=N}$	$R(N-1)$ variables
allocation distribution	$(\theta_i)_{i\in\mathbf{I}}$	$N-1$ variable

Table 2: Equilibrium variables

The number of variables exceeds the number of equations when:

$$S \leq R+1.$$

This holds regardless of the value for N^*, where $0 \leq N^* \leq N$.

A minor joint assumption on the Markov transition matrix Π and the endowments $(\mathbf{e}_i(s))_{(i,s)\in\mathbf{I}\times\mathbf{S}}$ is required to prove the result.

Assumption 3 For every $i \in \mathbf{I}$, $\exists s \in \mathbf{S}$ such that

$$\mathbf{e}_i(s) < \sum_{\sigma\in\mathbf{S}} \pi(s,\sigma) \left(\frac{\mathbf{E}(\sigma)}{\mathbf{E}(s)}\right)^{-\rho} \mathbf{e}_i(\sigma).$$

Assumption 3 is trivially satisfied for iid economies without aggregate risk as it simply implies that there exists at least 2 distinct endowment values in the set $(\mathbf{e}_i(s))_{s\in\mathbf{S}}$. For all other economies, Assumption 3 holds over a generic subset of the Markov probabilities and the household endowments.

Theorem 3 *Under Assumption 3, if $S \leq R+1$ (more variables than equations), then generically over the subset of household endowments $(\mathbf{e}_i(s))_{(i,s)\in\mathbf{I}\times\mathbf{S}}$, the $S(N+N^*-1)$ equations are linearly independent.*

Proof. See Section A.3. ■

The only economies consistent with the necessary condition in Theorem 3 and Assumption 1 are those with a full rank endowment matrix $R = N$ and $S = N+1$.

This prediction is in sharp contrast to those under inflation rate targeting and interest rate targeting. The reason is that symmetric policy under nominal GDP targeting does not reduce the rank of the payout matrix.

The condition $S = N+1$ is necessary, but is not sufficient, since the definition of a Markov equilibrium must additionally satisfy the following inequalities:

1. Monetary authority net domestic positions are nonnegative:

$$\hat{\mathbf{B}}_{i,i}(s) \geq 0 \text{ for all } i \in \mathbf{I}. \tag{34}$$

2. The return on money cannot dominate the return on 1-period nominal bonds:

$$\mathbf{q}_i(s) \leq 1 \text{ for all } (i,s) \in \mathbf{I} \times \mathbf{S}. \tag{35}$$

3. Strictly positive consumption:

$$\theta_i \in (0,1) \text{ for all } i \in \mathbf{I}. \tag{36}$$

5.5 Theoretical conclusions

Over a generic subset of household endowments $(\mathbf{e}_i(s))_{(i,s)\in\mathbf{I}\times\mathbf{S}}$, the following are not consistent with a Pareto efficient equilibrium allocation:

1. Symmetric policy, meaning all countries adopt the same type of targeting rule, with the exception of nominal GDP targeting when $S = N + 1$.
2. An exchange rate peg (since inflation rate targeting is not consistent with Pareto efficiency).
3. $S > N + 1$ under general policy, meaning all countries adopt some form of targeting rule, but not necessarily the same type of rule.

6 General Implications of Incomplete Markets

For both inflation rate targeting and interest rate targeting, no matter which targets are chosen, the rank of the real payout matrix equals 1. For the case of inflation rate targeting, the equilibrium allocation is equivalent to that obtained in a real economy with a risk-free real bond (with payouts proportional to $\overrightarrow{1}$).

The endowment matrix, a $S \times N$ matrix, contains the endowments for all states and all countries. We can assume that this matrix has full column rank N. Under nominal GDP targeting, the asset span, or the set of possible real portfolio payouts, is equal to the column space of the endowment matrix. If $\overrightarrow{1}$, the vector with risk-free payouts, is contained in this column span, then the allocation under nominal GDP targeting is constrained Pareto efficient, while the allocation under inflation rate targeting is constrained Pareto inefficient.

The vector $\overrightarrow{1}$ is contained in the column space of the endowment matrix if, for instance, the economy doesn't contain aggregate risk.

This means that the monetary authorities can change from inflation rate targeting to another policy and make some households better off without making others worse off. This is not possible under nominal GDP targeting as any change in policy must make some households worse off. The adjective 'constrained' refers to the fact that the asset structure, determined by the endowment matrix, is held fixed (with fewer assets than states of uncertainty). Policy chooses if the asset span is $1-$dimensional (in the case of inflation rate targeting) or is $N-$dimensional (in the case of nominal GDP targeting).

More generally, monetary policies don't have to follow targeting rules. Stationary policy only requires that monetary authorities target interest rates that are stationary (and only depend upon the current state realization). Under general policies, suppose that the real payout matrix has full column rank. The asset span is $N-$dimensional, but only under nominal GDP targeting is the asset span equal to the column space of the endowment matrix. Under other policies, the asset span will be a different linear subspace. Changes in the asset span under incomplete markets have real effects.

Allowing for general policies increases the number of policy choice variables, meaning that the necessary upper bound on S for Pareto efficiency is increased. Even so, provided that S is large enough, Pareto efficiency will not be consistent with equilibrium. Working in the set of Pareto inefficient equilibrium allocations, the asset span will be optimally chosen to maximize the designated social welfare function. The asset span will never be $1-$dimensional (inflation rate targeting, interest rate targeting, and exchange rate targeting), but will always be $N-$dimensional (with nominal GDP targeting serving as one of an infinite number of possible policies).

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A Appendix

A.1 Proof of Theorem 1

Consider the budget constraint for the household in country $i \in \mathbf{I}$ in date-event s^t :

$$c_i\left(s^t\right) + \sum_{j \in \mathbf{I}} q_j(s^t)\hat{b}_{i,j}(s^t) = \nu_i\left(s^t\right)\mathbf{e}_i\left(s_{t-1}\right) + \sum_{j \in \mathbf{I}} \nu_j\left(s^t\right)\hat{b}_{i,j}(s^{t-1}). \tag{37}$$

The exact same constraint can be written for date-event (s^t, σ) :

$$\left[c_i\left(s^t,\sigma\right) - \nu_i\left(s^t,\sigma\right)\mathbf{e}_i\left(s_t\right)\right] + \sum_{j \in \mathbf{I}} q_j(s^t,\sigma)\hat{b}_{i,j}(s^t,\sigma) = \sum_{j \in \mathbf{I}} \nu_j\left(s^t,\sigma\right)\hat{b}_{i,j}(s^t). \tag{38}$$

Multiply both sides of (38) by $\beta\left(\frac{c_i\left(s^t,\sigma\right)}{c_i\left(s^t\right)}\right)^{-\rho}$ and take the conditional expectation:

$$E_t\left[\beta\left(\frac{c_i\left(s^t,\sigma\right)}{c_i\left(s^t\right)}\right)^{-\rho}\left(c_i\left(s^t,\sigma\right) - \nu_i\left(s^t,\sigma\right)\mathbf{e}_i\left(s_t\right) + \sum_{j \in \mathbf{I}} q_j(s^t,\sigma)\hat{b}_{i,j}(s^t,\sigma)\right)\right] \tag{39}$$

$$= \sum_{j \in \mathbf{I}} \hat{b}_{i,j}(s^t)\left\{\beta\sum_{\sigma}\pi\left(s_t,\sigma\right)\left(\frac{c_i\left(s^t,\sigma\right)}{c_i\left(s^t\right)}\right)^{-\rho}\nu_j\left(s^t,\sigma\right)\right\}.$$

The Fisher equations from above imply that:

$$\beta \sum_{\sigma} \pi\left(s_t, \sigma\right) \left(\frac{c_i\left(s^t, \sigma\right)}{c_i\left(s^t\right)}\right)^{-\rho} \nu_j\left(s^t, \sigma\right) = q_j(s^t) \;\; \forall j \in \mathbf{I}.$$

This means that (39) is given by:

$$E_t\left[\beta \left(\frac{c_i\left(s^t, \sigma\right)}{c_i\left(s^t\right)}\right)^{-\rho} \left(\left(c_i\left(s^t, \sigma\right) - \nu_i\left(s^t, \sigma\right)\mathbf{e}_i\left(s_t\right) + \sum_{j \in \mathbf{I}} q_j(s^t, \sigma)\hat{b}_{i,j}(s^t, \sigma)\right)\right)\right] = \sum_{j \in \mathbf{I}} q_j(s^t)\hat{b}_{i,j}(s^t). \tag{40}$$

Inserting this new expression (40) back into the date-event s^t budget constraint (37) and iterating forward yields:

$$\nu_i\left(s^t\right)\mathbf{e}_i\left(s_{t-1}\right) + \sum_{j \in \mathbf{I}} \nu_j\left(s^t\right)\hat{b}_{i,j}(s^{t-1}) = \tag{41}$$

$$c_i\left(s^t\right) + \sum_{k=1}^{\infty} \beta^k E_t\left[\left(\frac{c_i\left(s^{t+k}\right)}{c_i\left(s^t\right)}\right)^{-\rho}\left(c_i\left(s^{t+k}\right) - \nu_i\left(s^{t+k}\right)\mathbf{e}_i\left(s_{t+k-1}\right)\right)\right],$$

after citing the transversality condition. The equilibrium equation (41) must hold in all date-events s^t.

If the equilibrium allocation is Pareto efficient, then Assumption 2 dictates that $c_i\left(s^t\right) = \theta_i\mathbf{E}\left(s_t\right)$. The equation (41) is updated as:

$$\nu_i\left(s^t\right)\mathbf{e}_i\left(s_{t-1}\right) + \sum_{j \in \mathbf{I}} \nu_j\left(s^t\right)\hat{b}_{i,j}(s^{t-1}) = \tag{42}$$

$$\theta_i\mathbf{E}\left(s_t\right) + \sum_{k=1}^{\infty} \beta^k E_t\left[\left(\frac{\mathbf{E}\left(s_{t+k}\right)}{\mathbf{E}\left(s_t\right)}\right)^{-\rho}\left(\theta_i\mathbf{E}\left(s_{t+k}\right) - \nu_i\left(s^{t+k}\right)\mathbf{e}_i\left(s_{t+k-1}\right)\right)\right].$$

The household wealth is defined by the left-hand side of (42):

$$\hat{\omega}_i(s^t) = \nu_i\left(s^t\right)\mathbf{e}_i\left(s_{t-1}\right) + \sum_{j \in \mathbf{I}} \nu_j\left(s^t\right)\hat{b}_{i,j}(s^{t-1}).$$

The right-hand side of (42), by definition, only depends upon the current state s_t. It does not depend upon any other realizations from the history s^{t-1}. Further, it does not depend upon any other equilibrium variables. The policy choice pins down the sequence $\left\{\nu_i\left(s^{t+k}\right)\right\}$. This verifies that the wealth $\hat{\omega}_i(s^t)$ only depends upon the current state realization s_t.

A.2 Proof of Theorem 2

The proof considers inflation rate targeting. The result for interest rate targeting proceeds in an identical manner.

Given that the rank of the payout matrix equals 1, each household and monetary authority will only hold the domestic bond (without loss of generality). The holdings for all foreign bonds will be set equal to 0.

The variables are

$$\xi = \left((\mu_i)_{i\in\mathbf{I}}, (\mathbf{b}_{i,i})_{i\in\mathbf{I}\setminus\{N\}}, (\mathbf{B}_{i,i})_{i\in\mathbf{I}}, (\theta_i)_{i\in\mathbf{I}} \right) \in \mathbb{R}^{2(2N-1)}$$

and the parameters are

$$\theta = (\mathbf{e}_i(s))_{(i,s)\in\mathbf{I}\times\mathbf{S}} \in \mathbb{R}^{SN}_{++}.$$

Define the system of equations as

$$\Phi : \mathbb{R}^{2(2N-1)+SN} \rightarrow \mathbb{R}^{S(2N-1)}$$

where

$$\Phi(\xi,\theta) = \begin{pmatrix} \left(\theta_i \mathbf{E}(s) - \mathbf{q}_i(s)\,\mathbf{e}_i(s) + \sum_{j\in\mathbf{I}} \mathbf{q}_j(s)\,\mathbf{b}_{i,j} - \sum_{j\in\mathbf{I}} \rho_j \mathbf{b}_{i,j} \right)_{(i,s)\in\mathbf{I}\setminus\{N\}\times\mathbf{S}} \\ \left(\mathbf{e}_i(s) - \mathbf{q}_i(s)\,\mathbf{e}_i(s) + \sum_{j\in\mathbf{I}} \mathbf{q}_j(s)\,\mathbf{B}_{i,j} - \sum_{j\in\mathbf{I}} \rho_j \mathbf{B}_{i,j} \right)_{(i,s)\in\mathbf{I}\times\mathbf{S}} \end{pmatrix}.$$

Notice that instead of SN household budget constraints and $S(N-1)$ monetary authority constraints, I chose to express the system equivalently as $S(N-1)$ household budget constraints and SN monetary authority constraints.

Define the projection $\phi : \mathbb{R}^{2(2N-1)+SN} \rightarrow \mathbb{R}^{SN}_{++}$ as the mapping $(\xi,\theta) \mapsto \theta$ such that $\Phi(\xi,\theta) = 0$. The mapping ϕ is proper iff for any compact subset of the range Y', the inverse image $\phi^{-1}(Y')$ is also compact.

Given the parametric transversality theorem, it suffices to prove that $D_{\xi,\theta}\Phi(\xi,\theta)$ has full row rank $S(2N-1)$. Specifically, for any $\alpha^T = \left(\left(\Delta b_i^T\right)_{i\in\mathbf{I}\setminus\{N\}}, \left(\Delta B_i^T\right)_{i\in\mathbf{I}} \right) \in \mathbb{R}^{S(2N-1)}$, the product

$$\alpha^T D_{\xi,\theta}\Phi(\xi,\theta) = 0 \tag{43}$$

implies $\alpha^T = 0$.

There exists more variables than equations (the necessary condition of Theorem 2) when

$$S \leq 2.$$

Choose country $i \in \mathbf{I} \backslash \{N\}$. Consider the columns for the derivatives with respect to $(\mathbf{b}_{i,i}, \theta_i)$. From (43), we have the following equations:

$$\Delta b_i^T \begin{bmatrix} (\mathbf{q}_i(1) - \rho_i) & \mathbf{E}(1) \\ (\mathbf{q}_i(2) - \rho_i) & \mathbf{E}(2) \end{bmatrix} = 0. \tag{44}$$

By definition, $\mathbf{q}_i(s) = \beta \rho_i \sum_{\sigma \in \mathbf{S}} \pi(s, \sigma) \left(\frac{\mathbf{E}(\sigma)}{\mathbf{E}(s)} \right)^{-\rho}$. This can be rewritten as:

$$(\mathbf{q}_i(s))_{s \in \mathbf{S}} = \beta \rho_i \hat{\Pi} \overrightarrow{1},$$

where the matrix $\hat{\Pi}$ has elements $\hat{\Pi}(s, \sigma) = \pi(s, \sigma) \left(\frac{\mathbf{E}(\sigma)}{\mathbf{E}(s)} \right)^{-\rho}$. The matrix in (44) can be written as:

$$\begin{bmatrix} \rho_i \left(\beta \hat{\Pi} - I_S \right) \overrightarrow{1} & (\mathbf{E}(s))_{s \in \mathbf{S}} \end{bmatrix}.$$

The matrix has full rank. Since it is square, then $\Delta b_i^T = 0$.

Choose country $i \in \mathbf{I}$. Consider the columns for the derivatives with respect to $\mathbf{B}_{i,i}$. With $S \leq 2$ and from (43), we have the following equations:

$$\Delta B_i^T \begin{bmatrix} (\mathbf{q}_i(1) - \rho_i) \\ (\mathbf{q}_i(2) - \rho_i) \end{bmatrix} = 0.$$

Thus, $\left(\Delta B_i^T \right)_{i \in \mathbf{I}}$ belong to a 1-dimensional linear subspace of $\mathbb{R}^2$. This implies that $\exists \delta \in \mathbb{R}$ such that $\Delta B_i^T(2) = \delta \Delta B_i^T(1)$ for all countries $i \in \mathbf{I}$.

Consider the columns for the derivatives with respect to $(\mathbf{e}_i(s))_{s \in \mathbf{S}}$. The equations (43) imply that:

$$\Delta B_i^T(s) (1 - \mathbf{q}_i(s)) = 0 \quad \forall s \in \mathbf{S}.$$

A.2.1 Subcase A

Suppose there exists some $s' \in \mathbf{S}$ such that $\mathbf{q}_i(s') < 1$. This dictates that $\Delta B_i^T(s') = 0$ and then $\Delta B_i^T(s) = 0 \ \forall s \in \mathbf{S}$.

Since (43) implies $\alpha^T = 0$, then $D_{\xi,\theta} \Phi(\xi, \theta)$ has full row rank.

A.2.2 Subcase B

Suppose that $\mathbf{q}_i(s) = 1\ \forall s \in \mathbf{S}$. The monetary authority constraints reduce to:

$$\left(\mathbf{q}_i(s) - \rho_i\right) \mathbf{B}_{i,i} = 0.$$

This equation can be equivalently written as:

$$\rho_i \left(\beta\hat{\Pi} - I_S\right) \overrightarrow{1}\ \mathbf{B}_{i,i} = 0.$$

This implies that $\mathbf{B}_{i,i} = 0$.

With $\mathbf{B}_{i,i} = 0$, then the monetary authority does not hold any debt positions. From the monetary authority constraint (21):

$$\nu_i\left(s^t\right) = \frac{\mathbf{e}_i(s_t)}{\mathbf{e}_i(s_{t-1})},$$

which contradicts that $\nu_i(s^t) = \rho_i$ for all date-events. Thus, Subcase B is not possible.

The analysis for interest rate targeting is identical.

A.3 Proof of Theorem 3

The variables are

$$\xi = \left((\mu_i)_{i\in\mathbf{I}^*}, (\mathbf{b}_{i,j})_{(i,j)\in\mathbf{I}\setminus\{N\}\times\mathbf{I}}, (\mathbf{B}_{i,j})_{(i,j)\in\mathbf{I}^*\times\mathbf{I}}, (\theta_i)_{i\in\mathbf{I}}\right) \in \mathbb{R}^{(N+N^*-1)(R+1)}$$

and the parameters are

$$\theta = (\mathbf{e}_i(s))_{(i,s)\in\mathbf{I}\times\mathbf{S}} \in \mathbb{R}^{SN}_{++}.$$

Define the system of equations as

$$\Phi : \mathbb{R}^{(N+N^*-1)(R+1)+SN} \longrightarrow \mathbb{R}^{S(N+N^*-1)}$$

where

$$\Phi(\xi,\theta) = \begin{pmatrix} \left(\theta_i \mathbf{E}(s) + \sum_{j\in\mathbf{I}} \mathbf{q}_j(s)\,\mathbf{e}_j(s)\mathbf{b}_{i,j} - \mu_i \mathbf{e}_i(s) - \sum_{j\in\mathbf{I}} \mu_j \mathbf{e}_j(s)\mathbf{b}_{i,j}\right)_{(i,s)\in\mathbf{I}\setminus\{N\}\times\mathbf{S}} \\ \left(\mathbf{e}_i(s) + \sum_{j\in\mathbf{I}} \mathbf{q}_j(s)\,\mathbf{e}_j(s)\mathbf{B}_{i,j} - \mu_i \mathbf{e}_i(s) - \sum_{j\in\mathbf{I}} \mu_j \mathbf{e}_j(s)\mathbf{B}_{i,j}\right)_{(i,s)\in\mathbf{I}^*\times\mathbf{S}} \end{pmatrix}.$$

Define the projection $\phi : \mathbb{R}^{(N+N^*-1)(R+1)+SN} \longrightarrow \mathbb{R}^{SN}_{++}$ as the mapping $(\xi, \theta) \mapsto \theta$ such that $\Phi(\xi, \theta) = 0$. The mapping ϕ is proper iff for any compact subset of the range Y', the inverse image $\phi^{-1}(Y')$ is also compact.

A.3.1 Subcase A

Suppose that $[\nu']$ has full rank. If $[\nu']$ has full rank, then ϕ is proper.

Given the parametric transversality theorem, it suffices to prove that $D_{\xi,\theta}\Phi(\xi, \theta)$ has full row rank $S(N+N^*-1)$. Specifically, for any $\alpha^T = \left(\left(\Delta b_i^T\right)_{i\in\mathbf{I}\backslash\{N\}}, \left(\Delta B_i^T\right)_{i\in\mathbf{I}^*} \right) \in \mathbb{R}^{S(N+N^*-1)}$, the product

$$\alpha^T D_{\xi,\theta}\Phi(\xi, \theta) = 0 \tag{45}$$

implies $\alpha^T = 0$.

When $rank\,[\nu'] = R = N$, then there exists more variables than equations (the necessary condition of Theorem 2) when

$$S \leq N + 1.$$

Choose country $i \in \mathbf{I}\backslash\{N\}$. Consider the columns for the derivatives with respect to $\left((\mathbf{b}_{i,j})_{j\in\mathbf{I}}, \theta_i \right)$. From (45), we have the following equations:

$$\Delta b_i^T \begin{bmatrix} \mathbf{e}_1(1)(\mathbf{q}_1(1) - \mu_1) & \mathbf{e}_N(1)(\mathbf{q}_N(1) - \mu_N) & \mathbf{E}(1) \\ \vdots & \vdots & \vdots \\ \mathbf{e}_1(S)(\mathbf{q}_1(S) - \mu_1) & \mathbf{e}_N(S)(\mathbf{q}_N(S) - \mu_N) & \mathbf{E}(S) \end{bmatrix} = 0. \tag{46}$$

By definition, $\mathbf{q}_i(s) = \beta\mu_i \sum_{\sigma\in\mathbf{S}} \pi(s, \sigma) \left(\frac{\mathbf{E}(\sigma)}{\mathbf{E}(s)}\right)^{-\rho} \frac{\mathbf{e}_i(\sigma)}{\mathbf{e}_i(s)}$. This can be rewritten as:

$$(\mathbf{q}_i(s)\,\mathbf{e}_i(s))_{s\in\mathbf{S}} = \beta\mu_i\hat{\Pi}\,(\mathbf{e}_i(\sigma))_{\sigma\in\mathbf{S}},$$

where the matrix $\hat{\Pi}$ has elements $\hat{\Pi}(s, \sigma) = \pi(s, \sigma)\left(\frac{\mathbf{E}(\sigma)}{\mathbf{E}(s)}\right)^{-\rho}$. If $[\nu']$ has full rank, then the matrix $\begin{bmatrix} \mathbf{e}_1(1) & .. & \mathbf{e}_N(1) \\ \vdots & .. & \vdots \\ \mathbf{e}_1(S) & .. & \mathbf{e}_N(S) \end{bmatrix}$ has full rank. The first N columns of the matrix in (46) are equivalently expressed as:

$$\left[\mu_1 \left(\beta\hat{\Pi} - I_S\right)(\mathbf{e}_1(\sigma))_{\sigma\in\mathbf{S}} \quad \mu_N \left(\beta\hat{\Pi} - I_S\right)(\mathbf{e}_N(\sigma))_{\sigma\in\mathbf{S}} \right], \tag{47}$$

where I_S is the $S-$dimensional identity matrix. The matrix $\left(\beta\hat{\Pi} - I_S\right)$ has full rank, since

$\beta \in (0,1)$ and Π is a transition matrix. The final column in the matrix in (46) is linearly independent from any of the first N columns, provided that $\mathbf{q}_j(s) = \mathbf{q}_j(s')$ for some (j, s, s').

Suppose, in order to obtain a contradiction, that $\exists j \in \mathbf{I}$ such that $\mathbf{q}_j(s) = \mathbf{q}_j(1)$ for all $s \in \mathbf{S}$. Given the nominal GDP targeting rules, the Euler equations (27) in matrix form are given by:

$$(\mathbf{e}_j(\sigma))_{\sigma\in\mathbf{S}} = \frac{\mu_j \beta}{\mathbf{q}_j(1)} \hat{\Pi} (\mathbf{e}_j(\sigma))_{\sigma\in\mathbf{S}}.$$

This implies that:

$$\left[I - \frac{\mu_j \beta}{\mathbf{q}_j(1)} \hat{\Pi} \right] (\mathbf{e}_j(\sigma))_{\sigma\in\mathbf{S}} = 0. \tag{48}$$

The matrix $I - \frac{\mu_j\beta}{\mathbf{q}_j(1)}\hat{\Pi}$ has full rank provided that $\frac{\mu_j\beta}{\mathbf{q}_j(1)} < 1$. The term $\frac{\mu_j\beta}{\mathbf{q}_j(1)} \geq 1$ iff

$$\frac{\mu_j\beta}{\mathbf{q}_j(1)} = \frac{\mathbf{e}_i(s)}{\sum_{\sigma\in\mathbf{S}} \pi(s,\sigma) \left(\frac{\mathbf{E}(\sigma)}{\mathbf{E}(s)}\right)^{-\rho} \mathbf{e}_i(\sigma)} \geq 1$$

for all states $s \in \mathbf{S}$. As this contradicts Assumption 3, then $I - \frac{\mu_j\beta}{\mathbf{q}_j(1)}\hat{\Pi}$ has full rank and equation (48) implies $(\mathbf{e}_j(\sigma))_{\sigma\in\mathbf{S}} = 0$, a contradiction. This finishes the claim that $\forall j$, $\exists s, s'$ such that $\mathbf{q}_j(s) = \mathbf{q}_j(s')$.

The matrix in (46) is a full rank and square matrix. Thus, $\Delta b_i^T = 0$.

Choose country $i \in \mathbf{I}^*$. Consider the columns for the derivatives with respect to $(\mathbf{B}_{i,j})_{j\in\mathbf{I}}$. With $S \leq N+1$ and from (45), we have the following equations:

$$\Delta B_i^T \begin{bmatrix} \mathbf{e}_1(1)(\mathbf{q}_1(1) - \mu_1) & \mathbf{e}_N(1)(\mathbf{q}_N(1) - \mu_N) \\ : & : \\ \mathbf{e}_1(S)(\mathbf{q}_1(S) - \mu_1) & \mathbf{e}_N(S)(\mathbf{q}_N(S) - \mu_N) \end{bmatrix} = 0. \tag{49}$$

The matrix $\begin{bmatrix} \mathbf{e}_1(1)(\mathbf{q}_1(1) - \mu_1) & \mathbf{e}_N(1)(\mathbf{q}_N(1) - \mu_N) \\ : & : \\ \mathbf{e}_1(S)(\mathbf{q}_1(S) - \mu_1) & \mathbf{e}_N(S)(\mathbf{q}_N(S) - \mu_N) \end{bmatrix}$, as previously shown, has full column rank $N = S - 1$. This implies that $\left(\Delta B_i^T\right)_{i\in\mathbf{I}}$ belong to a 1-dimensional linear subspace of $\mathbb{R}^S$. This implies that $\exists (\delta_s)_{s\in\mathbf{S}} \in \mathbb{R}^S$ such that $\Delta B_j(s) = \delta_s \Delta B_j(1)$ for all countries $j \in \mathbf{I}^*$ and for all states $s \in \mathbf{S}$.

For any country $j \in \mathbf{I}^*$, consider the columns for the derivatives with respect to $(\mathbf{e}_j(s))_{s\in\mathbf{S}}$. The equations (45) imply that:

$$\mathbf{q}_j(s) - \mu_j \left(\sum_{i\in\mathbf{S}} \Delta B_i(s) \mathbf{B}_{i,j} + \Delta B_j(s) \left(1 - \mu_j\right) \right) = 0 \quad \forall (j,s) \in \mathbf{I}^* \times \mathbf{S}. \tag{50}$$

Using the result that $\Delta B_j(s) = \delta_s \Delta B_j(1)$ for all countries $j \in \mathbf{I}^*$ and for all states $s \in \mathbf{S}$, then (50) is equivalently given by (as the terms δ_s cancel):

$$\mathbf{q}_j(s) - \mu_j \sum_{i \in \mathbf{S}} \Delta B_i(1) \mathbf{B}_{i,j} + \Delta B_j(1) \left(1 - \mu_j\right) = 0 \quad \forall (j,s) \in \mathbf{I}^* \times \mathbf{S}. \tag{51}$$

This implies that $\forall s, s' \in \mathbf{S}$,

$$\mathbf{q}_j(s) \sum_{i \in \mathbf{S}} \Delta B_i(1) \mathbf{B}_{i,j} = \mathbf{q}_j(s') \sum_{i \in \mathbf{S}} \Delta B_i(1) \mathbf{B}_{i,j},$$

as this is the only term that contains s.

From the claim above, $\forall j, \exists s, s'$ such that $\mathbf{q}_j(s) = \mathbf{q}_j(s')$. This implies that $\sum_{i \in \mathbf{S}} \Delta B_i(1) \mathbf{B}_{i,j} = 0$. From (51), this implies $\Delta B_j(1) \left(1 - \mu_j\right) = 0$. Since $\mu_j = 1$ for $j \in \mathbf{I}^*$, then $\Delta B_j(1) = 0$. Since $\Delta B_j(s) = \delta_s \Delta B_j(1)$ for all countries $j \in \mathbf{I}^*$ and for all states $s \in \mathbf{S}$, then $\Delta B_j^T = 0$.

Therefore, $\alpha^T = \left(\left(\Delta b_i^T \right)_{i \in \mathbf{I} \setminus \{N\}}, \left(\Delta B_i^T \right)_{i \in \mathbf{I}^*} \right) = 0$. Since (45) implies $\alpha^T = 0$, then the matrix $D_{\xi,\theta} \Phi(\xi, \theta)$ has full row rank.

A.3.2 Subcase B

Suppose that $[\nu']$ does not have full rank, with $rank\,[\nu'] = R < N$. Define the variables to only include R bonds with linearly independent payouts. The payout matrix associated with these R bonds has full rank. The number of equations remains the same: $S(N + N^* - 1)$. There exists more variables than equations when

$$S \leq R + 1.$$

This is the necessary condition of Theorem 3. The exact same argument as in Subcase B remains valid.